# Lecture 17. Magnetic fields in matter II.
## The auxiliary field H

**Last time:**

→ Types of magnetic materials

* diamagnetic → magnetization opposite to external $\underline{B}$
  ↳ tilt of $e^-$ orbits

* paramagnetic → magnetization // to external $\underline{B}$
  ↳ originates from the spin of $e^-$

* ferromagnetic → Retain magnetization in the absence of external $\underline{B}$
  ↳ Magnetization depends on history of material

→ Magnetic dipoles

$\underline{m} = I\underline{a}$ ; Force on a dipole $\underline{F} = \nabla(\underline{m}\cdot\underline{B})$

torque on a dipole $\underline{\tau} = \underline{m} \times \underline{B}$

* Magnetization

$$M \equiv \frac{\text{magnetic dipole moment}}{\text{unit volume}} \quad ; \quad \underline{M} = \frac{1}{V}\sum_i m_i$$

* Bound currents

$\underline{K}_b = \underline{M}\times\hat{n}$ bound surface current

$\underline{J}_b = \nabla\times\underline{M}$ bound volume current

**Ampère's law for magnetized materials**

$\underline{J} = \underline{J}_b + \underline{J}_f$ total current is equal to the bound current + free current

$\underline{J}_b$: due to magnetization

$\underline{J}_f$: transport of charge

In terms of the total current we write ampère's law:

$$\nabla\times\underline{B} = \mu_0\underline{J} = \mu_0(\underline{J}_b + \underline{J}_f)$$

$$\Rightarrow \frac{1}{\mu_0}\nabla\times\underline{B} = (\underbrace{\nabla\times\underline{M}}_{\text{by definition } \underline{J}_b} + \underline{J}_f)$$

$$\Rightarrow \nabla \times \underbrace{\left(\frac{1}{\mu_0}\underline{B} - \underline{M}\right)}_{\equiv H \text{ auxiliary field}} = \underline{J}_f$$

With this definition

$$\nabla \times \underline{H} = \underline{J}_f$$

$$\oint \underline{H} \cdot d\underline{l} = I_{f\,enc}$$

; with $\underline{H} \equiv \frac{1}{\mu_0}\underline{B} - \underline{M}$

↑ plays the same role as $\underline{D}$ in electrostatics.

Remember that $\underline{H} \neq \underline{B}$

we know that the div of $\underline{B}$ is zero, this is not the case for $\underline{H}$:

$$\nabla \cdot \underline{H} = \nabla \cdot \left(\frac{1}{\mu_0}\underline{B} - \underline{M}\right) = -\nabla \cdot \underline{M}$$

Example: A long copper rod of radius R carries a uniformly distributed free current I. Find $\underline{H}$ inside & outside the rod.


I is longitudinal

$\Rightarrow \underline{B}$ will be circumferential

In which direction will $\underline{M}$ be?

copper is weakly diamagnetic

we will have a bound current in the opposite direction as I

we will use Ampère's law for $\underline{H}$:

$$\oint \underline{H} \cdot d\underline{l} = I_{f\,enc}$$

We draw an amperian loop

For a loop of radius $s < R$

$$\oint \underline{H} \cdot d\underline{l} = \underline{H} \int d\underline{l} = \underline{H}\,(2\pi s) = I_{f\,enc} \quad (*)$$

If the current is uniformly distributed

$$\underline{J} = \frac{I}{\pi R^2} \qquad \underline{J} = \frac{\text{current}}{\text{unit area}}$$

In the amperian loop the current enclosed will be proportional for the area of the loop:

$$I_{fenc} = J(\pi s^2) = \frac{I}{\pi R^2}(\pi s^2) = \frac{I s^2}{R^2}$$

so subs this $I_{fenc}$ in eq (*):

$$\underline{H}(2\pi s) = \frac{I s^2}{R^2}$$

$$\Rightarrow \underline{H} = \frac{I}{2\pi R^2} s \hat{\Phi}$$ for $s < R$ inside the wire

For outside the wire, the current enclosed is the total current.

$$H(2\pi s) = I_{enc} = I$$


$$\Rightarrow \underline{H} = \frac{I}{2\pi s}\hat{\Phi}$$ for $s \geq R$ for points outside of the wire.

In the outside region $\underline{M} = 0$

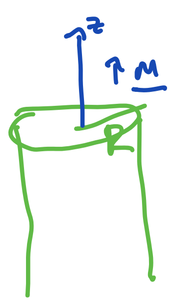
$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \cancel{\underline{M}}^{\,0} = \frac{1}{\mu_0}\underline{B} = \frac{I}{2\pi s} \Rightarrow \boxed{\underline{B} = \frac{\mu_0 I}{2\pi s}\hat{\Phi}}$$

For inside the wire, we don't know $\underline{M}$, so we don't know $\underline{B}$.

EXAMPLE: An infinitely long cylinder of radius R, carries a frozen in magnetization $\underline{M}$ parallel to the axis

$\underline{M} = ks\hat{z}$ where k is a constant
S is the distance from the axis.

There is no free current anywhere.
Find the magnetic field inside and outside the cylinder.




1) We will find $\underline{J}_b$ and $\underline{K}_b$ and calculate $\underline{B}$

2) We will use Ampère's law for $\underline{H}$ and calculate $\underline{B}$

1) The formulas for the bound currents are:

$$\underline{K}_b = \underline{M} \times \hat{n}$$

$$\underline{J}_b = \nabla \times \underline{M}$$

We have a cylinder so it is best to use cylindrical coord. For any vector $\underline{A}$:

$$\nabla \times \underline{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{z}$$

[Appendix F Purcell p. 742)

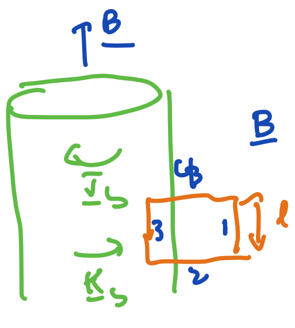
First we'll calculate $\underline{J}_b$

$$\underline{J}_b = \nabla \times \underline{M} = \frac{1}{s}\left[\frac{\partial}{\partial \theta}(ks)\right)\Big]\hat{r} - \frac{\partial}{\partial s}(ks)\,\hat{\theta} = -k\hat{\theta}$$

Now we calculate $\underline{K}_b$; we notice $\hat{n}$ is the radial direction.

$$\underline{K}_b = \underline{M} \times \hat{n} = \begin{array}{ccc} \hat{r} & \hat{\theta} & \hat{z} \\ \end{array}$$
$$\begin{vmatrix} 0 & 0 & kR \\ 1 & 0 & 0 \end{vmatrix} = kR\hat{\theta}$$

$$\therefore \underline{J}_b = -k\hat{\theta} \quad \text{and} \quad \underline{K}_b = kR\hat{\theta}$$




$\underline{B}$ is in the $\hat{z}$ direction → given by the right hand rule.

Outside $\underline{B} = 0$

For the region inside the cylinder:

$$\oint \underline{B}\cdot d\underline{l} = \underbrace{\int_1 \underline{B}\cdot d\underline{l}}_{\substack{\text{outside}\\ =0}} + \cancel{\int_2 \underline{B}\cdot d\underline{l}} + \underbrace{\int_3 \underline{B}\cdot d\underline{l}}_{= Bl} + \cancel{\int_4 \underline{B}\cdot d\underline{l}} = \mu_0 I_{enc}$$

sides 2 & 4 = 0 because they are ⊥ to $\underline{B}$. In region 1 $\underline{B} = 0$ because there is no current.

Using Ampère's law and the loop $\square$:

$$B = \frac{\mu_0 I_{enc}}{\ell}$$ what is $I_{enc}$? It's the sum of the bound currents.

$$I_J = \int \underline{J}\cdot d\underline{a} \quad \text{and} \quad I_K = \int \underline{K}\cdot d\underline{l} \quad ; \quad I_{enc} = I_J + I_K$$

$$\Rightarrow I_{enc} = \int \underline{J}_b \cdot d\underline{a} + \int \underline{K}_b \cdot d\underline{l} = \int -k\,da + \int kR\,dl = -k\int da + kR\int dl$$

$$= -k[l(R-s)] + kRl = -kRl + kls + kRl$$

$$= kls$$

so the magnetic field is

$$B = \frac{\mu_0}{\ell}(kls) = \mu_0 ks \Rightarrow \boxed{\underline{B} = \mu_0 ks\hat{z}}$$ inside

2) We will use Ampère's law and $\underline{H}$ to calculate $\underline{B}$:

$$\oint \underline{H}\cdot d\underline{l} = I_{f\,enc}$$

We use the same loop


$$I_{f\,enc} = 0 \qquad \Rightarrow \oint \underline{H}\cdot d\underline{l} = Hl = 0 \Rightarrow \underline{H} = 0$$

And

$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M} = 0 \quad (*)$$

Outside the cylinder $\underline{M} = 0 \Rightarrow \underline{B} = 0$

Inside the cylinder

$\underline{M} = ks\hat{z}$ subst this in (*)

$$\frac{1}{\mu_0}\underline{B} = ks\hat{z} \Rightarrow \boxed{\underline{B} = \mu_0 ks\hat{z}}$$ inside

## Linear media

For most materials $\underline{M}$ is proportional $\underline{B}$:

In the case of $\underline{H}$, we can write this proportionality

$$\underline{M} = \chi_m \underline{H} \quad ; \; \chi_m \equiv \text{magnetic susceptibility}$$

its dimensionless

it's + for paramagnets

it's − for diamagnets.

We can rewrite this last eq in terms of $\underline{B}$:

$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M} \Rightarrow \underline{B} = \mu_0(\underline{H} + \underline{M}) = \mu_0(\underline{H} + \chi_m \underline{H})$$

$$= \underbrace{\mu_0(1+\chi_m)}_{\equiv \mu \text{ permeability of the material}}\underline{H}$$

$$\Rightarrow \underline{B} = \mu \underline{H} \quad \text{where} \quad \mu = \mu_0(1+\chi_m) \text{ permeability.}$$

Even for linear media $\nabla \cdot \underline{H} \neq 0$; $\underline{B}$ and $\underline{H}$ are NOT equivalent

$$\nabla \cdot \underline{H} = \nabla \cdot \left(\frac{1}{\mu}\underline{B}\right) = \frac{1}{\mu}\nabla \cdot \underline{B} + \underline{B}\,\nabla\left(\frac{1}{\mu}\right) = \underline{B} \cdot \left(\frac{1}{\mu}\right)$$

if $\mu$ is changing

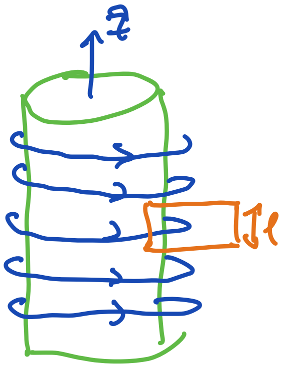
$\nabla \cdot \underline{H} \neq 0$

For example at the boundary of the material $\mu$ will change.

For linear materials the bound current is proportional to the free current:

$$\underline{J}_b = \nabla \times \underline{M} = \nabla \times (\chi_m \underline{H}) = \chi_m \underline{J}_f$$

Example: An infinite solenoid ($n$ turns/unit length, current $I$) is filled with linear material of susceptibility $\chi_m$. Find the magnetic field inside the solenoid




We will use ampère's law for $\underline{H}$.

$$\oint \underline{H} \cdot d\underline{l} = I_{fenc} \Rightarrow H \oint d\underline{l} = I_{fenc}$$

this reduces to calculating $\oint d\underline{l}$ along a loop and finding $I_{fenc}$

$\oint d\underline{l} = l$ because outside $\underline{M} = 0 \Rightarrow H = 0$

what is the free current?

$I_{fenc} = NI$ where $N$ is the total # of turns.

$$Hl = NI \Rightarrow \underline{H} = \frac{NI}{l}\hat{z} = nI\hat{z}$$

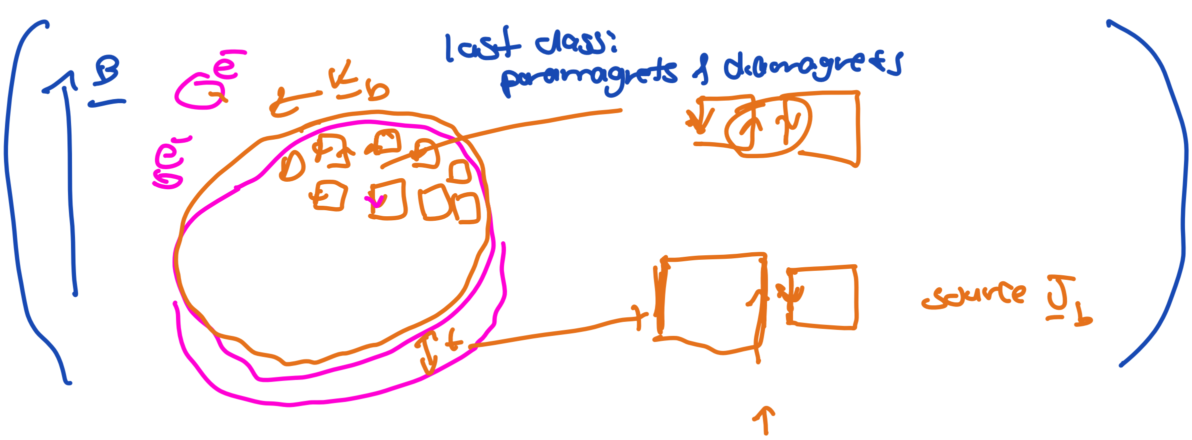
$n = \frac{N}{l}$ $\frac{\#\text{ turns}}{\text{unit length}}$.

Since we have a linear material $\underline{B} = \mu \underline{H}$

$$\Rightarrow \underline{B} = \mu \underline{H} = \mu_0 (1 + \chi_m) nI\hat{z}$$

Ferromagnets

Like paramagnetism & diamagnetism ferromagnetism involves the alignment of magnetic dipoles. Dipoles are associated with the spins of unpaired electrons.



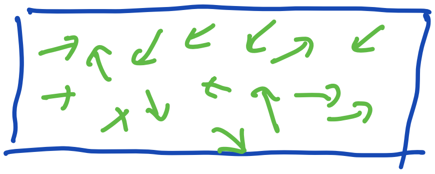
# Ferromagnets

Apply an external $\underline{B}$

or

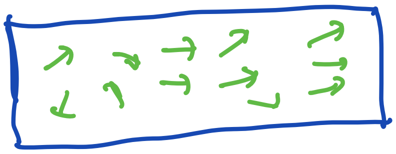
lower the temperature.

a



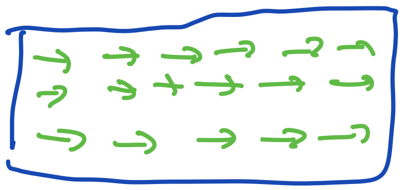
unmagnetized material
domains randomly oriented
w/ respect to each other

c



slightly magnetized
material

b



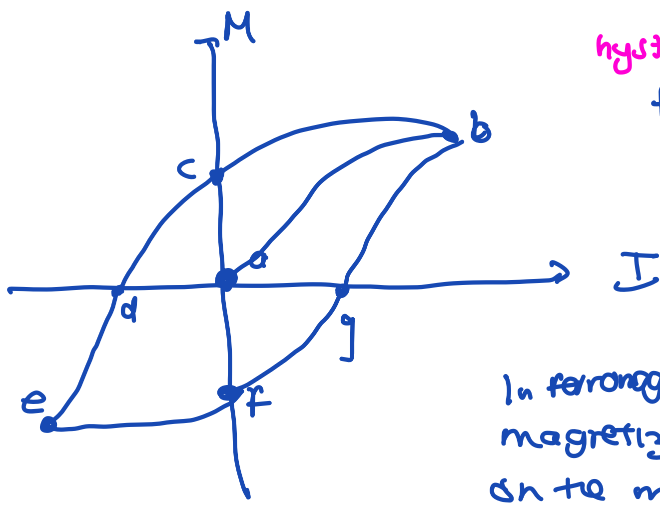
strongly magnetized
material

① Quantum effects → Spins need to align to lower the energy of the system

② Long range effect: Magnetization will produce a field that align to away.

③ Reducing temp → Reduces thermal motions, there will be spontaneous lining up of spins without external $\underline{B}$.

The temperature at which a material becomes ferromagnetic is known as the curie point.




hysteresis loop
from the greek
"lag behind"

In ferromagnets the magnetization depends on the material's history.

electromagnet